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Hyers–Ulam–Rassias stability of Cauchy equation in the space of Schwartz distributions ☆

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Abstract

We reformulate and prove the Hyers–Ulam–Rassias stability of Cauchy equation in the space of Schwartz tempered distributions and Fourier hyperfunctions.

Keywords: Gauss transforms; Heat kernel; Distributions; Hyperfunctions; Heat polynomial; Stability; Cauchy equation

1. Introduction

Generalizing the stability theorem of D.H. Hyers [10,11] which was motivated by S.M. Ulam [17], Th.M. Rassias [15] and Z. Gajda [7] showed the following stability theorem for the Cauchy equation:

Theorem 1.1 [7,15]. *Let $f : E_1 \to E_2$ with E_1, E_2 Banach spaces be an approximately additive, that is, f satisfies*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon\bigl(\|x\|^p + \|y\|^p\bigr), \quad p \neq 1, \tag{1.1}$$

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for all $x, y \in E_1$ ($x \neq 0$ *and* $y \neq 0$ *if* $p < 0$). *Then there exists a unique mapping* $g: E_1 \to E_2$ *such that*

$$g(x+y) - g(x) - g(y) = 0$$

and

$$\| f(x) - g(x) \| \leqslant \frac{2\epsilon}{|2^p - 2|} \|x\|^p \tag{1.2}$$

for all $x \in E_1$. *Here the inequality* (1.1) *holds for* $x, y \neq 0$ *and* (1.2) *for* $x \neq 0$ *if* $p < 0$.

The above stability theorem was firstly proved for the case $p < 1$ by Th.M. Rassias [15] and later it was proved for the case $p > 1$ by Z. Gajda [7].

In this paper, we reformulate and prove the above stability theorem in the spaces of generalized functions such as the space $\mathcal{S}'$ of Schwartz tempered distributions and the space $\mathcal{F}'$ of Fourier hyperfunctions for the case that p is even integer grater than 2. Note that the above inequality (1.1) makes no sense if f is a tempered distribution or Fourier hyperfunction. As in [1,2,5,6,13] making use of the pullbacks of generalized function we extend the inequality (1.1) to the spaces of tempered distributions and Fourier hyperfunctions as follows:

$$\|u \circ A - u \circ P_1 - u \circ P_2\| \leqslant \epsilon \bigl(\|x\|^{2p} + \|y\|^{2p}\bigr). \tag{1.3}$$

Here $u \circ A$, $u \circ P_1$ and $u \circ P_2$ are the pullbacks of u in $\mathcal{S}'$ or $\mathcal{F}'$ by A, P_1 and P_2, respectively, where A, P_1 and P_2 are the functions $A(x, y) = x + y$, $P_1(x, y) = x$ and $P_2(x, y) = y$, $x, y \in \mathbb{R}^n$. Also the inequality $\|v\| \leqslant \epsilon(\|x\|^{2p} + \|y\|^{2p})$ in (1.3) means that $|\langle v, \varphi\rangle| \leqslant \epsilon \int (\|x\|^{2p} + \|y\|^{2p}) |\varphi(x, y)|\, dx\, dy$ for all test functions $\varphi \in \mathcal{S}$ (respectively $\mathcal{F}$).

For the pullback of tempered distributions we refer to [9, Chapters V–VI]. As a matter of fact, the pullbacks $u \circ A$, $u \circ P_1$, $u \circ P_2$ can be written in a transparent way as

$$\bigl\langle u \circ A, \varphi(x, y)\bigr\rangle = \left\langle u, \int \varphi(x - y, y)\, dy \right\rangle,$$

$$\bigl\langle u \circ P_1, \varphi(x, y)\bigr\rangle = \left\langle u, \int \varphi(x, y)\, dy \right\rangle,$$

$$\bigl\langle u \circ P_2, \varphi(x, y)\bigr\rangle = \left\langle u, \int \varphi(x, y)\, dx \right\rangle$$

for all test functions $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$.

As a results, we prove that every solution of the inequality (1.3) can be approximated by a linear function in the sense that there exists a unique $a \in \mathbb{C}^n$ such that

$$\|u - a \cdot x\| \leqslant \frac{2\epsilon}{4^p - 2} \|x\|^{2p}.$$

2. Distributions and hyperfunctions

We first introduce briefly some spaces of generalized functions such as the space $\mathcal{S}'$ of tempered distributions and the space $\mathcal{F}'$ of Fourier hyperfunctions which is a natural generalization of $\mathcal{S}'$. Here we use the multi-index notations for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$,

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \qquad \alpha! = \alpha_1! \ldots \alpha_n!,$$
$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \qquad \partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n},$$

where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \partial/\partial x_j$.

Definition 2.1 [3,8,9,16]. We denote by $\mathcal{S}$ or $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{\alpha,\beta} = \sup_x \left| x^\alpha \partial^\beta \varphi(x) \right| < \infty \tag{2.1}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. The elements of $\mathcal{S}$ are called rapidly decreasing functions and the elements of the dual space $\mathcal{S}'$ are called *tempered distributions*.

As a matter of fact, it is known in [3] that (2.1) is equivalent to

$$\sup_{x \in \mathbb{R}^n} \left| x^\alpha \varphi(x) \right| < \infty, \qquad \sup_{\xi \in \mathbb{R}^n} \left| \xi^\beta \hat{\varphi}(\xi) \right| < \infty \tag{2.1$'$}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$.

Imposing growth conditions on $\|\cdot\|_{\alpha,\beta}$ in (2.1) Sato and Kawai introduced the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions as follows:

Definition 2.2 [4,8,16]. We denote by $\mathcal{F}$ or $\mathcal{F}(\mathbb{R}^n)$ the Sato space of all infinitely differentiable functions φ in $\mathbb{R}^n$ such that

$$\|\varphi\|_{A,B} = \sup_{x,\alpha,\beta} \frac{|x^\alpha \partial^\beta \varphi(x)|}{A^{|\alpha|} B^{|\beta|} \alpha! \beta!} < \infty \tag{2.2}$$

for some positive constants A, B.

We say that $\varphi_j \to 0$ as $j \to \infty$ if $\|\varphi_j\|_{A,B} \to 0$ as $j \to \infty$ for some $A, B > 0$, and denote by $\mathcal{F}'$ the strong dual of $\mathcal{F}$ and call its elements *Fourier hyperfunctions*.

It is known in [4] that the inequality (2.2) is equivalent to

$$\sup_{x \in \mathbb{R}^n} \left| \varphi(x) \right| \exp k|x| < \infty, \qquad \sup_{\xi \in \mathbb{R}^n} \left| \hat{\varphi}(\xi) \right| \exp h|\xi| < \infty \tag{2.2$'$}$$

for some $h, k > 0$.

It is easy to see the following topological inclusions:

$$\mathcal{F} \hookrightarrow \mathcal{S}, \qquad \mathcal{S}' \hookrightarrow \mathcal{F}'.$$

From now on *a test function* means an element in the Schwartz space $\mathcal{S}$ or the Sato space $\mathcal{F}$ and *a generalized function* means *a tempered distribution* or *a Fourier hyperfunction*.

3. Main theorems

We employ the n-dimensional heat kernel, that is, the fundamental solution $E_t(x)$ of the heat operator $\partial_t - \Delta_x$ in $\mathbb{R}^n_x \times \mathbb{R}^+_t$ given by

$$E_t(x) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t), & t > 0, \\ 0, & t \leqslant 0. \end{cases}$$

Here the semigroup property

$$(E_t * E_s)(x) = E_{t+s}(x) \tag{3.1}$$

of the heat kernel will be very useful later. Now let a tempered distribution u be given. Then its *Gauss transform*

$$Gu(x,t) = (u * E_t)(x) = \langle u_y, E_t(x-y)\rangle, \quad x \in \mathbb{R}^n,\ t > 0, \tag{3.2}$$

is a C^∞-function in $\mathbb{R}^n \times (0,\infty)$. As a matter of fact we can represent tempered distributions via some solutions of the heat equation as follows:

Proposition 3.1 [14]. *Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then its Gauss transform $Gu(x,t)$ is a C^∞-solution of heat equation satisfying*:

(i) *There exist positive constants C, M and N such that*

$$\bigl|Gu(x,t)\bigr| \leqslant Ct^{-M}\bigl(1+|x|\bigr)^N \quad \text{in } \mathbb{R}^n \times (0,\delta); \tag{3.3}$$

(ii) *$Gu(x,t) \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in \mathcal{S}$,*

$$\langle u, \varphi\rangle = \lim_{t\to 0^+} \int Gu(x,t)\varphi(x)\,dx.$$

Conversely, every C^∞-solution $U(x,t)$ of heat equation satisfying the growth condition (3.3) *can be uniquely expressed as $U(x,t) = Gu(x,t)$ for some $u \in \mathcal{S}'$.*

Similarly we can represent Fourier hyperfunctions as initial values of solutions of heat equation as a special case of the results in [12]. In this case, the estimate (3.3) is replaced by the following: For every $\epsilon > 0$ there exists a positive constant C_ϵ such that

$$\bigl|Gu(x,t)\bigr| \leqslant C_\epsilon \exp\bigl(\epsilon\bigl(|x| + 1/t\bigr)\bigr) \quad \text{in } \mathbb{R}^n \times (0,\delta). \tag{3.3$'$}$$

Definition 3.2. Let v be in $\mathcal{S}'$ or $\mathcal{F}'$. Then we denote by $\|v\| \leqslant \psi$ if

$$\bigl|\langle v, \varphi\rangle\bigr| \leqslant \|\psi\varphi\|_{L^1} \tag{3.4}$$

for all test functions φ.

Now we prove main theorems.

Theorem 3.3. *Let u in $\mathcal{S}'$ or $\mathcal{F}'$ satisfy the inequality*

$$\|u \circ A - u \circ P_1 - u \circ P_2\| \leqslant \epsilon(x^{2\gamma} + y^{2\gamma}) \tag{3.5}$$

for some $\gamma \in \mathbb{N}_0^n$, $|\gamma| > 1$. *Then there exists a unique* $a \in \mathbb{C}^n$ *such that*

$$\|u - a \cdot x\| \leqslant \frac{2\epsilon}{4^{|\gamma|} - 2} x^{2\gamma}. \tag{3.6}$$

Proof. Convolving in each side of (3.5) the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels as a function of x, y the left-hand side of (3.5) can be written as

$$\begin{aligned}
\big[(u \circ A) * \big(E_t(x)E_s(y)\big)\big](\xi, \eta) &= \langle u \circ A, E_t(\xi - x)E_s(\eta - y)\rangle \\
&= \left\langle u_x, \int E_t(\xi - x + y)E_s(\eta - y)\,dy\right\rangle \\
&= \left\langle u_x, \int E_t(\xi + \eta - x - y)E_s(y)\,dy\right\rangle \\
&= \langle u_x, (E_t * E_s)(\xi + \eta - x)\rangle \\
&= \langle u_x, E_{t+s}(\xi + \eta - x)\rangle \\
&= Gu(\xi + \eta, t + s),
\end{aligned}$$

and similarly

$$\big[(u \circ P_1) * \big(E_t(x)E_s(y)\big)\big](\xi, \eta) = Gu(\xi, t),$$
$$\big[(u \circ P_2) * \big(E_t(x)E_s(y)\big)\big](\xi, \eta) = Gu(\eta, s),$$

where $Gu(\xi, t)$ is the Gauss transform of u.

Also the right-hand side of (3.5) can be written as

$$\big[\epsilon(x^{2\gamma} + y^{2\gamma}) * \big(E_t(x)E_s(y)\big)\big](\xi, \eta) = \epsilon\big(H_{2\gamma}(\xi, t) + H_{2\gamma}(\eta, s)\big),$$

where $H_{2\gamma}$ is the heat polynomial of degree 2γ which is given by

$$H_{2\gamma}(\xi, t) = \big[x^{2\gamma} * E_t(x)\big](\xi) = (2\gamma)! \sum_{0 \leqslant \alpha \leqslant \gamma} \frac{t^{|\alpha|}\xi^{2\gamma - 2\alpha}}{\alpha!(2\gamma - 2\alpha)!}.$$

Thus the inequality (3.5) is converted to the following stability problem involving the Gauss transform of u:

$$\big|Gu(\xi + \eta, t + s) - Gu(\xi, t) - Gu(\eta, s)\big| \leqslant \epsilon\big(H_{2\gamma}(\xi, t) + H_{2\gamma}(\eta, s)\big) \tag{3.7}$$

for all $\xi, \eta \in \mathbb{R}^n$, $t, s > 0$.

Now we follow the same method as in [15]. Replacing both ξ and η by $\xi/2$, both t and s by $t/2$ in (3.7) we have

$$\big|Gu(\xi, t) - 2Gu(2^{-1}\xi, 2^{-1}t)\big| \leqslant 2\epsilon H_{2\gamma}(2^{-1}\xi, 2^{-1}t)$$

for all $\xi \in \mathbb{R}^n$, $t > 0$. Making use of the induction argument and triangle inequality we have

$$\begin{aligned}
\big|Gu(\xi, t) - 2^n Gu(2^{-n}\xi, 2^{-n}t)\big| &\leqslant \epsilon \sum_{j=1}^{n} 2^j H_{2\gamma}(2^{-j}\xi, 2^{-j}t) \\
&\leqslant \epsilon(2\gamma)! \sum_{0 \leqslant \alpha \leqslant \gamma} a_{n,\alpha} \frac{t^{|\alpha|}\xi^{2\gamma - 2\alpha}}{\alpha!(2\gamma - 2\alpha)!}
\end{aligned} \tag{3.8}$$

for all $n \in \mathbb{N}$, $\xi \in \mathbb{R}^n$, $t > 0$, where $a_{n,\alpha} = 2^{|\alpha|+1}(1 - 2^{(|\alpha|-|2\gamma|+1)n})/(2^{|2\gamma|} - 2^{|\alpha|+1})$.

Replacing ξ, t by $2^{-m}\xi, 2^{-m}t$, respectively in (3.8) and multiplying 2^m in the result it follows easily from the fact $|\gamma| > 1$ that

$$A_m(\xi, t) := 2^n Gu(2^{-m}\xi, 2^{-m}t)$$

is a Cauchy sequence which converges locally uniformly. Now let

$$A(\xi, t) = \lim_{m\to\infty} A_m(\xi, t).$$

Then $A(\xi, t)$ is the unique mapping in $\mathbb{R}^n \times (0, \infty)$ satisfying

$$\left|Gu(\xi, t) - A(\xi, t)\right| \leqslant \epsilon(2\gamma)! \sum_{0\leqslant\alpha\leqslant\gamma} a_\alpha \frac{t^{|\alpha|}\xi^{2\gamma-2\alpha}}{\alpha!(2\gamma - 2\alpha)!}, \tag{3.9}$$

$$A(\xi + \eta, t + s) - A(\xi, t) - A(\eta, s) = 0 \tag{3.10}$$

for all $\xi, \eta \in \mathbb{R}^n$, $t, s > 0$, where $a_\alpha = 2^{|\alpha|+1}/(2^{|2\gamma|} - 2^{|\alpha|+1})$. Indeed, the inequality (3.9) follows immediately from (3.8). To prove (3.10), replacing ξ, η, t, s by $2^{-m}\xi, 2^{-m}\eta, 2^{-m}t, 2^{-m}s$ in (3.7), respectively, multiplying 2^m and letting $m \to \infty$ it follows immediately from the fact $|\gamma| > 1$. To prove the uniqueness of $A(\xi, t)$, let $B(\xi, t)$ be another function satisfying (3.9) and (3.10). Then it follows from (3.9), (3.10) and the triangle inequality that for all $n \in \mathbb{N}$,

$$\begin{aligned}\left|A(\xi, t) - B(\xi, t)\right| &\leqslant n\left|A\left(\frac{x}{n}, \frac{t}{n}\right) - B\left(\frac{x}{n}, \frac{t}{n}\right)\right| \\ &\leqslant 2\epsilon(2\gamma)! n^{1-|\gamma|} \sum_{0\leqslant\alpha\leqslant\gamma} a_\alpha \frac{t^{|\alpha|}\xi^{2\gamma-2\alpha}}{\alpha!(2\gamma - 2\alpha)!}.\end{aligned} \tag{3.11}$$

Letting $n \to \infty$, we have $A(\xi, t) = B(\xi, t)$ for all $\xi \in \mathbb{R}^n$, $t > 0$. This proves the uniqueness.

Now it is easy to see that every continuous solution $A(\xi, t)$ of the Cauchy equation (3.10) has the form

$$A(\xi, t) = a \cdot \xi + bt$$

for some $a \in \mathbb{C}^n$, $b \in \mathbb{C}$. Letting $t \to 0^+$ in (3.9) we have

$$\|u - a \cdot \xi\| \leqslant \frac{2\epsilon}{4^{|\gamma|} - 2}\xi^{2\gamma}.$$

This completes the proof. □

Now we consider the inequality

$$\|u \circ A - u \circ P_1 - u \circ P_2\| \leqslant \epsilon\big(\|x\|^{2p} + \|y\|^{2p}\big)$$

for some integer $p > 1$.

Theorem 3.4. *Let u in $\mathcal{S}'$ or $\mathcal{F}'$ satisfy the inequality*

$$\|u \circ A - u \circ P_1 - u \circ P_2\| \leqslant \epsilon\big(\|x\|^{2p} + \|y\|^{2p}\big) \tag{3.12}$$

for some integer $p > 1$. *Then there exists a unique* $a \in \mathbb{C}^n$ *such that*

$$\|u - a \cdot x\| \leqslant \frac{2\epsilon}{4^p - 2}\|x\|^{2p}. \tag{3.13}$$

Proof. Note that we can write

$$\|x\|^{2p} = \sum_{|\gamma|=p} \frac{p!}{\gamma!} x^{2\gamma}.$$

Thus convolving in each side of (3.12) the tensor product $E_t(x)E_s(y)$ of n-dimensional heat kernels as a function of x, y the inequality (3.12) is converted to the following inequality as in the proof of Theorem 3.3:

$$\begin{aligned} &\bigl|Gu(\xi + \eta, t + s) - Gu(\xi, t) - Gu(\eta, s)\bigr| \\ &\quad \leqslant \epsilon \sum_{|\gamma|=p} \frac{p!}{\gamma!}\bigl(H_{2\gamma}(\xi, t) + H_{2\gamma}(\eta, s)\bigr) \end{aligned} \tag{3.14}$$

for all $\xi, \eta \in \mathbb{R}^n$, $t, s > 0$.

Now making use of the same approach as in the proof of above theorem we have

$$\|u - a \cdot \xi\| \leqslant \sum_{|\gamma|=p} \frac{p!}{\gamma!}\left(\frac{2\epsilon}{4^{|\gamma|} - 2}\xi^{2\gamma}\right) = \frac{2\epsilon}{4^p - 2}\|\xi\|^{2p}.$$

This completes the proof. $\square$

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